Topological Black Holes in Quantum Gravity

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Abstract

We derive the black hole solutions with horizons of non-trivial topology and investigate their properties in the framework of an approach to quantum gravity being an extension of Bohm's formulation of quantum mechanics. The solutions we found tend asymptotically (for large r) to topological black holes. We also analyze the thermodynamics of these space-times.

1   Introduction

In the recent paper [1] we started preliminary investigations of black hole solutions of quantum gravity. In this paper we would like to extend this investigations to the case of the so–called topological black holes, being generalizations of the standard Schwarzschild black hole solution of general relativity, characterized by non-trivial topology of the horizons. The constant time sections of such space-times are of the generic form

$$ds_3^2 = A(r)dr^2 + r^2 d^2\Omega, \tag{1}$$

where $d^2\Omega$ is a metric of a Riemann surface of genus $1, 0, -1$, which in coordinates θ, ϕ reads

$$d^2\Omega = d\theta^2 + \sin^2\theta d\phi^2, \quad k = 1, \quad \text{sphere};$$

$$d^2\Omega = d\theta^2 + d\phi^2, \quad k = 0, \quad \text{torus};$$

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$$d^2\Omega = d\theta^2 + \sinh^2\theta d\phi^2, \quad k = -1, \quad \text{pseudo-sphere}.$$

Such space-times and their thermodynamics have been analyzed recently from many perspectives in papers [2], [3], [4], [5], [6].

In this paper we would like to investigate properties of topological black holes in the context of the formalism of approaching quantum gravity developed in [7]. This formalism has its roots in the David Bohm's approach to quantum mechanics (see e.g., [8] and [9]) and has been extended to the case of quantum gravity in [10] and [11] (in minisuperspace) and in [12], [13] (for full theory.)

The starting point of our investigation will be the Hamiltonian constraint of the theory, which is the Hamiltonian constraint of general relativity modified by "quantum potential" (for derivation of this formula, see [7]):

$$0 = \mathcal{H}_\perp = \kappa^2 G_{abcd} p^{ab} p^{cd} +$$

$$\mathcal{F}\left(\frac{27}{16} \frac{\rho^{(5)2}\kappa^2}{\Lambda^2}\sqrt{h} + \frac{1}{\kappa^2}\sqrt{h}R \right.$$

$$\left. -\frac{8}{9}\frac{\Lambda^2}{\kappa^6\rho^{(5)2}}\sqrt{h}\left(-\frac{3}{8}R^2 + R_{ab}R^{ab} \right) \right), \tag{2}$$

$$\mathcal{F} = \frac{1}{2}\frac{\sin^2(\phi)}{\left\{ \cosh\left(\frac{3\rho^{(5)}}{\Lambda}\mathcal{V} + \frac{4\Lambda}{3\kappa^4\rho^{(5)}}\mathcal{R} \right) + \cos(\phi) \right\}^2}, \tag{3}$$

where κ is the gravitational constant, G_{abcd} is the Wheeler–De Witt metric, $\rho^{(5)}$ is the renormalization

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constant, Λ is the bare cosmological and ϕ is a parameter characterizing the solution of Wheeler–De Witt equation, the following formulas are based on.

2   Solutions

We are interested in the static case, where momenta are equal to zero. In this case one of the dynamical equation of the theory (corresponding to the 00 component of Einstein field equations) is the requirement that the Hamiltonian constraint with $p^{ab} = 0$ vanishes, to wit

$$\frac{27}{16}\frac{\rho^{(5)2}\kappa^2}{\Lambda^2}\sqrt{h} + \frac{1}{\kappa^2}\sqrt{h}R$$
$$-\frac{8}{9}\frac{\Lambda^2}{\kappa^6\rho^{(5)2}}\sqrt{h}\left(-\frac{3}{8}R^2 + R_{ab}R^{ab}\right) = 0 \qquad (4)$$

It is worth observing that the cosmological and renormalization constants appear only in combination $v_0 = \frac{\Lambda}{\rho^{(5)}}$ of dimension $[m]^3$. Thus the theory possesses two length scales, the Planck scale l defined by κ, and $v_0^{1/3}$.

For the metric of (1), the solution takes the form

$$A = (k + f(r)r^2)^{-1}, \quad k = 1, 0, -1 \qquad (5)$$

where

$$f(r) = p \pm \sqrt{\frac{3}{4}p^2 + \frac{\alpha}{r^3}}, \qquad (6)$$

and $p = \frac{9}{4}\frac{l^4}{v_0^2}$ is a dimensionful parameter, whose interpretation will be found below. To find the space-time metrics we make the anzatz

$$ds^2 = -N(r)dt^2 + ds_3^2$$

and make use of the following variational principle, resulting from the ADM action in the gauge where the shift vector $N^a = 0$,

$$I = -\frac{1}{4}\int N\mathcal{H}_\perp.$$

As a result we find

$$ds^2 = -A(r)^{-1}dt^2 + A(r)dr^2 + d\Omega^2. \qquad (7)$$

3   Properties of the metric

The metric (7) should reduce to the metric of the Einstein topological black hole in the limit $r \to \infty$. Indeed, one would expect that the quantum gravity modifications should be small at large distances. This is indeed the case. For large r we have

$$r^2 f(r) \to r^2 p\left(1 \pm \frac{\sqrt{3}}{2}\right) \pm \frac{2\alpha}{3pr}. \qquad (8)$$

For topological black hole we have

$$A(r) = \left(k - \frac{2M}{r} - \frac{\lambda r^2}{3}\right)^{-1},$$

and thus asymptotically, our solution corresponds to the topological black hole in the Anti de Sitter space with cosmological constant

$$\lambda = -3p\left(1 \pm \frac{\sqrt{3}}{2}\right) \qquad (9)$$

and the mass

$$M = \mp\frac{2\alpha}{3p}. \qquad (10)$$

We see therefore that the parameter p is to be interpreted (up to the numerical factor) as the physical cosmological constant.

In what follows we will consider only the case of positive M and α. It can be checked that in the other case (upper sign), the space time develops a circular singularity at $r = \sqrt[3]{\frac{3p^2}{4|\alpha|}}$. Moreover in this case there is no horizon for $k = 0, 1$. We will investigate such a situation in a separate paper.

Let us now turn to horizons of our black holes. They are given as zeros of the function $A^{-1}(r)$. In the case $k = 0$ the position of horizon can easily be computed explicitly, to wit

$$r_h = \sqrt[3]{\frac{4\alpha}{p^2}} \qquad (11)$$

The relation

$$r_h \sim \alpha^{1/3}$$

holds also in the cases $k = \pm 1$ for large values of α. The Killing vector $\partial/\partial t$ is timelike for $r > r_h$ and

spacelike for $r < r_h$. In particular, the singularity at $r = 0$ is spacelike as in the Schwartzschild black hole.

In the case $k = +1$ there are two horizons at r_-, r_h, corresponding to two real positive roots of the equation $A(r) = 0$. There is only one solution of this equation for $\alpha = \alpha_{crit} = \sqrt{\frac{80\,p}{27} + \frac{56\,\sqrt{7}\,p}{27}}$ and no horizon for smaller α.

In the case $k = -1$ we have one horizon at $r = r_h$ for all values of α. It is clear that

$$r_h > r_{crit} = \frac{1}{\sqrt{p}}\sqrt{\frac{2}{2 - \sqrt{3}}},$$

which is the minimal horizon radius, corresponding to $\alpha = 0$.

Thus the singularity is timelike for $k = 1$ and spacelike for $k = -1$ and is naked for $k = 1$ and $\alpha < \alpha_{crit}$.

4   Thermodynamics

For metrics of the form (7) the standard procedure leads to the expression for temperature

$$T = \frac{1}{4\pi}\left(\frac{\partial A^{-1}(r)}{\partial r}\right)_{r=r_h}. \qquad (12)$$

Substituting (5, 6) we find

$$T = \frac{r_h}{4\pi}\left(2\,p - \frac{\sqrt{3\,p^2 + \frac{4\,\alpha}{r_h^3}}\,\left(\alpha + 3\,p^2\,r_h^3\right)}{4\,\alpha + 3\,p^2\,r_h^3}\right), \qquad (13)$$

and asymptotically for large masses the relation $T \sim r_h$ i.e., $\alpha \sim T^3$ holds.

Let us now calculate the entropy. We will proceed exactly as in [1]. The Euclidean action for our solution is free energy divided by temperature

$$I_E = \frac{M}{T} - S. \qquad (14)$$

It is well known that I_E consists of bulk integral and the boundary terms B at infinity and at $r = r_h$ which are fixed by boundary conditions of the variational problem for the action I_E. Thus we consider

$$I_E = -\int_{r_+}^{\infty} N_0 F' + B,$$

where

$$F(r) = \frac{1}{4l^2 p}\frac{(kA-1)^2}{A^2 r} + \frac{r}{2l^2}\frac{kA-1}{A} + \frac{p}{8l^2}r^3.$$

The integral is a linear combination of constraints $\mathcal{H}_\perp(r)$, and thus

$$B = \frac{M}{T} - S. \qquad (15)$$

Now it follows that

$$B = \frac{M}{T} + 4\pi\int dr_+ \left(\frac{\partial F}{\partial A^{-1}(r)}\right)_{r=r_+} - S_0, \qquad (16)$$

where S_0 is a constant to be fixed in a moment. The entropy of the black hole of outer radius r_h is therefore equal

$$S = -4\pi\int dr_+ \left(\frac{\partial F}{\partial A^{-1}(r)}\right)_{r=r_+} + S_0 =$$

$$= k\frac{2\pi}{9l^2 p}\log r_h + \frac{\pi}{l^2}r_h^2 + S_0. \qquad (17)$$

For $k = 0$ the temperature vanishes for $r_h = 0$ and the logarithmic term in (17) is not present. Therefore we put $S_0 = 0$ in this case, and the entropy is purely of the Beckenstein–Hawking form.

In the case $k = -1$ situation is less clear. Let us observe that for the minimal radius of the horizon $r_{crit} = \frac{1}{\sqrt{p}}\sqrt{\frac{2}{2 - \sqrt{3}}}$, the temperature does not vanish. Nevertheless it seems reasonable to assume that in this case the temperature

$$T(r_{crit}) = \frac{r_{crit}\,p}{4\pi}\left(2 - \sqrt{3}\right)$$

is the minimal temperature and to take S_0 so that

$$S^{(-1)} = -\frac{2\pi}{9l^2 p}\log\left(\frac{r_h}{r_{crit}}\right) + \frac{\pi}{l^2}\left(r_h^2 - r_{crit}^2\right), \qquad (18)$$

as in the case $k = 1$, [1]. It can be easily checked that so defined entropy is always positive.

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